

# The adjacency matrix of one type of graph and the Fibonacci numbers

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## Abstract

Recently there is huge interest in graph theory and intensive study on computing integer powers of matrices.

In this paper, we investigate relationships between one type of graph and well-known Fibonacci sequence. In this content, we consider the adjacency matrix of one type of graph with  $2k$  ( $k = 1, 2, \dots$ ) vertices. It is also known that for any positive integer  $r$ , the  $(i, j)$ th entry of  $A^r$  ( $A$  is the adjacency matrix of the graph) is just the number of walks from vertex  $i$  to vertex  $j$ , that use exactly  $k$  edges.

Keywords: Fibonacci number, adjacency matrix, eigenvalue

## 1 Introduction

There are many special types of matrices which have great importance in many scientific work. For example matrices of tridiagonal, pentadiagonal and others. These types of matrices frequently appear in interpolation, numerical analysis, solution of boundary value problems, high order harmonic spectral filtering theory and so on. In [4]-[6], the authors investigated computing integer powers of some type of these matrices.

Among numerical sequences, the Fibonacci numbers which is defined by the recurrence  $F_{n+2} = F_{n+1} + F_n$  for  $n \geq 0$ , with initial conditions  $F_0 = 0$  and  $F_1 = 1$ , has achieved a kind of celebrity status. Although Fibonacci sequence has been studied extensively for hundreds of years, it remains to fascinating and there always seems to be some amazing properties aspects that are revealed by looking at it closely [1]. Fibonacci sequence has many applications in diverse fields such as mathematics, computer science, physics, biology and statistics.

A graph  $G = (V, E)$  occurs from two finite sets, with set of vertices  $V(G) = \{1, 2, \dots, n\}$  and set of edges  $E(G) = \{e_1, e_2, \dots, e_m\}$ . Let  $G$  be a

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graph with vertices  $v_1, v_2, \dots, v_n$ . The adjacency matrix of  $G$  is an  $n$ -square matrix  $A$  whose  $(i, j)$ th entry, denoted by  $[A]_{i,j}$ , is defined by:

$$[A]_{i,j} = \begin{cases} 1, & \text{if } v_i \text{ and } v_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Also, it is known that, for any positive integer  $r$ , the  $(i, j)$  entry of  $A^r$  is equal to the number of walks from  $v_i$  to  $v_j$  that use exactly  $r$  edges [3].

In [7], the authors consider the number of independent sets in graphs with two elementary cycles. They described the extremal values of the number of independent sets using Fibonacci and Lucas numbers.

In [8], the author investigated the relationship between  $k$ -Lucas sequence and 1-factors of a bipartite graph.

In [9], the authors consider a new family of  $k$ -Fibonacci numbers and investigate some properties of the relation and well-known Fibonacci numbers. They give the well-known Binet's formula

$$F_n = \frac{1}{\sqrt{5}}(\alpha^{n+1} - \beta^{n+1})$$

where  $\alpha = (1 + \sqrt{5})/2$  and  $\beta = (1 - \sqrt{5})/2$ .

In [10], the authors give a generalization for known-sequences and then they give the graph representations of the sequences. They generalize Fibonacci, Lucas, Pell and Tribonacci numbers and they show that the sequences are equal to the total number of  $k$ -independent sets of special graphs.

In [11], the author derived an explicit formula which corresponds to the Fibonacci numbers for the number of spanning trees given below:

Figure 1:

In this paper, we consider one type of graph with  $2k$  vertices ( $n = 2k$ ,  $k = 1, 2, 3, \dots, \frac{n}{2}$ ), each of its connected components of length 2 and every vertex has degree 1 or 3. The components of length 2 are all in the form an edge followed by a loop. Clearly:

Figure 1

The adjacency matrix of the graph given in Figure 1 is an  $n$ -square  $(0, 1)$ -block-diagonal matrix whose diagonal blocks have the form  $[0, 1, 1, 1]$ , which is:

$$A = \begin{cases} a_{i,i+1} = a_{i+1,i} = 1, & \text{for } i = 1, 3, 5, \dots, n-1 \\ a_{i,i} = 1, & \text{for } i = 2, 4, 6, \dots, n \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

The  $(i, j)$ th entry of  $A^r$  is just the number of the different paths from vertex  $i$  to vertex  $j$ . In other words, the number of the different paths from vertex  $i$  to vertex  $j$  corresponds Fibonacci numbers.

## 2 Main results

Let us consider the adjacency matrix of the graph given as in (1). One can observe that all integer powers of  $A$  are specified to the famous Fibonacci numbers with positive and negative signs.

It is also known that the  $r$ th ( $r \in \mathbb{N}$ ) power of a matrix is computed by using the known expression  $A^r = T J^r T^{-1}$  [2], here  $J$  is the Jordan form of the matrix and  $T$  is the transforming matrix. The matrices  $J$  and  $T$  are obtained using eigenvalues and eigenvectors of the matrix  $A$ . The eigenvalues of  $A$  are the roots of the characteristic equation defined by  $|A - \lambda I| = 0$  where  $I$  is the identity matrix of  $n$ th order.

Let  $P_n(x)$  be the characteristic polynomial of the matrix  $A$  which is defined in (1). Then we can write:

$$\begin{aligned} P_2(x) &= x^2 - x - 1 \\ P_4(x) &= x^4 - 2x^3 - x^2 + 2x + 1 \\ P_6(x) &= x^6 - 3x^5 + 5x^3 - 3x - 1 \\ P_8(x) &= x^8 - 4x^7 + 2x^6 + 8x^5 - 5x^4 - 8x^3 + 2x^2 + 4x + 1 \\ &\vdots \end{aligned} \quad (2)$$

Taking (2) into account

$$\begin{aligned} P_n(\lambda) &= (\lambda^2 - \lambda - 1)^{\frac{n}{2}} \\ &= [(\lambda - \alpha)(\lambda - \beta)]^{\frac{n}{2}} \end{aligned}$$

where  $n = 2k$ , ( $k = 1, 2, \dots$ ),  $\alpha = \frac{1 + \sqrt{5}}{2}$  and  $\beta = \frac{1 - \sqrt{5}}{2}$ . The eigenvalues of the matrix are multiple according to the order of the matrix  $A$ . Then the Jordan's form of the matrix  $A$  is:

$$J = J_k = \underbrace{\text{diag}(\alpha, \dots, \alpha)}_{k \text{ times}} \underbrace{\text{diag}(\beta, \dots, \beta)}_{k \text{ times}} \quad (3)$$

where  $k = 1, 2, 3, \dots, \frac{n}{2}$ . Let us consider the relation  $J = T^{-1}AT$  ( $AT = TJ$ ); here  $A$  is  $n$ th order matrix ( $n = 2k$ ,  $k = 1, 2, \dots$ ),  $J$  is the jordan form of the matrix  $A$  and  $T$  is the transforming matrix. We will find the transforming matrix  $T$ . Let us denote the  $j$ -th column of  $T$  by  $T_j$ . Then  $T = (T_1, T_2, \dots, T_n)$  and

$$(AT_1, \dots, AT_n) = (\lambda_1 T_1, \dots, \lambda_1 T_k, \lambda_2 T_{k+1}, \dots, \lambda_2 T_{2k}).$$

In other words

$$\begin{aligned} AT_1 &= \lambda_1 T_1 \\ AT_2 &= \lambda_1 T_2 \\ &\vdots \\ AT_k &= \lambda_1 T_k \\ AT_{k+1} &= \lambda_2 T_{k+1} \\ AT_{k+2} &= \lambda_2 T_{k+2} \\ &\vdots \\ AT_{2k} &= \lambda_2 T_{2k} \end{aligned} \quad (4)$$

Solving the set of equations system, we obtain the eigenvectors of the matrix  $A$ :

$$T = \left( \underbrace{\begin{pmatrix} 0 & 0 & \dots & 1 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & \alpha & 0 & 0 & \dots & 0 & \beta \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \beta & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & \alpha & \dots & 0 & 0 & \beta & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 & 0 \\ \alpha & 0 & \dots & 0 & \beta & 0 & \dots & 0 & 0 \end{pmatrix}}_k \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & \beta \end{pmatrix}}_k \right) \quad (5)$$

We shall find the inverse matrix  $T^{-1}$  denoting the  $i$ th row of the inverse matrix  $T^{-1}$  by  $T^{-1} = (t_1, t_2, \dots, t_n)$  and implementing the necessary transformations, we obtain:

$$T^{-1} = \frac{1}{\alpha - \beta} \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 & 0 & -\beta & 1 \\ 0 & 0 & 0 & 0 & \dots & -\beta & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & -\beta & 1 & \dots & 0 & 0 & \alpha & -1 \\ -\beta & 1 & 0 & 0 & \dots & \alpha & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \alpha & -1 & \dots & 0 & 0 & 0 & 0 \\ \alpha & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \end{pmatrix} \quad (6)$$

Using the equalities (3), (5) and (6); we derive the expression for the  $r$ th power of the matrix  $A$  :

$$A = TJT^{-1} \Rightarrow A^r = TJ^rT^{-1} = [a_{i,j}(r)]_{n \times n} \quad (7)$$

That is,

$$A^r = \begin{cases} a_{i-1,i-1}(r) = \frac{1}{\alpha - \beta}(-\beta\alpha^r + \alpha\beta^r) \\ a_{i,i}(r) = \frac{1}{\alpha - \beta}(\alpha^{r+1} - \beta^{r+1}) \\ a_{i-1,i}(r) = \frac{1}{\alpha - \beta}(\alpha^r - \beta^r) \\ a_{i,i-1}(r) = \frac{1}{\alpha - \beta}(-\beta\alpha^{r+1} + \alpha\beta^{r+1}) \\ 0, \quad \text{otherwise} \end{cases}$$

where  $i = 2, 4, 6, \dots, n$ .

**Lemma 1** Let  $A$  be as in (1). Then

$$\det(A) = (-1)^k$$

where  $n = 2k, k = 1, 2, \dots, \frac{n}{2}$ .

**Proof.** Using Laplace expansion, the determinant can be obtained. ■

**Corollary 2** Let  $A = [a_{ij}]$  be  $n$ -square matrix as in (1). Negative integer powers are:

$$A^{-r} = \begin{cases} a_{i-1,i-1}(-r) = \frac{1}{\alpha - \beta}((- \beta)^{r+1} - (-\alpha)^{r+1}) \\ a_{i,i}(-r) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^r + \alpha(-\beta)^r) \\ a_{i-1,i}(-r) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^{r+1} + \alpha(-\beta)^{r+1}) \\ a_{i,i-1}(-r) = \frac{1}{\alpha - \beta}((- \beta)^r - (-\alpha)^r) \\ 0, \quad \text{otherwise} \end{cases} \quad (8)$$

where  $i = 2, 4, 6, \dots, n$  and  $r = 1, 2, \dots$

**Proof.** By (7), we can write  $J = T^{-1}AT$ . We also can rewrite

$$J^{-1} = (T^{-1}AT)^{-1} = T^{-1}A^{-1}T.$$

It is known that the jordan matrix for  $A^{-1}$  is:

$$J^{-1} = \underbrace{\text{diag}(1/\alpha, \dots, 1/\alpha)}_{k \text{ times}} \underbrace{\text{diag}(1/\beta, \dots, 1/\beta)}_{k \text{ times}} \quad (9)$$

Provided the equalities (5), (6) and (9), the proof can be seen easily. ■

### 3 Examples

We can find the arbitrary integer powers of the matrix  $A$ , taking into account derived expressions. For example, if  $k = 3$ :

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

For  $r = 4$ :

$$A^4 = \begin{cases} a_{11}(4) = a_{33}(4) = a_{55}(4) = \frac{1}{\alpha - \beta}(-\beta\alpha^4 + \alpha\beta^4) = F_3 \\ a_{22}(4) = a_{44}(4) = a_{66}(4) = \frac{1}{\alpha - \beta}(\alpha^5 - \beta^5) = F_5 \\ a_{12}(4) = a_{34}(4) = a_{56}(4) = \frac{1}{\alpha - \beta}(\alpha^4 - \beta^4) = F_4 \\ a_{21}(4) = a_{43}(4) = a_{65}(4) = \frac{1}{\alpha - \beta}(-\beta\alpha^5 + \alpha\beta^5) = F_4 \\ 0, \quad \text{otherwise} \end{cases}$$

For  $r = -4$ :

$$A^{-4} = \begin{cases} a_{11}(-4) = a_{33}(-4) = a_{55}(-4) = \frac{1}{\alpha - \beta}((-\beta)^5 - (-\alpha)^5) = F_5 \\ a_{22}(-4) = a_{44}(-4) = a_{66}(-4) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^4 + \alpha(-\beta)^4) = F_3 \\ a_{12}(-4) = a_{34}(-4) = a_{56}(-4) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^5 + \alpha(-\beta)^5) = -F_4 \\ a_{21}(-4) = a_{43}(-4) = a_{65}(-4) = \frac{1}{\alpha - \beta}((-\beta)^4 - (-\alpha)^4) = -F_4 \\ 0, \quad \text{otherwise} \end{cases}$$

For  $r = -5$ :

$$A^{-5} = \begin{cases} a_{i-1,i-1}(-5) = \frac{1}{\alpha - \beta}((-\beta)^6 - (-\alpha)^6) = -F_6 \\ a_{i,i}(-5) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^5 + \alpha(-\beta)^5) = -F_4 \\ a_{i-1,i}(-5) = \frac{1}{\alpha - \beta}(-\beta(-\alpha)^6 + \alpha(-\beta)^6) = F_5 \\ a_{i,i-1}(-5) = \frac{1}{\alpha - \beta}((-\beta)^5 - (-\alpha)^5) = F_5 \\ 0, \quad \text{otherwise} \end{cases}.$$

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